

# Independence number of generalized Petersen graphs

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## Abstract

Determining the size of a maximum independent set of a graph  $G$ , denoted by  $\alpha(G)$ , is an NP-hard problem. Therefore many attempts are made to find upper and lower bounds, or exact values of  $\alpha(G)$  for special classes of graphs.

This paper is aimed toward studying this problem for the class of generalized Petersen graphs. We find new upper and lower bounds and some exact values for  $\alpha(P(n, k))$ . With a computer program we have obtained exact values for each  $n < 78$ . In [2] it is conjectured that the size of the minimum vertex cover,  $\beta(P(n, k))$ , is less than or equal to  $n + \lceil \frac{n}{5} \rceil$ , for all  $n$  and  $k$  with  $n > 2k$ . We prove this conjecture for some cases. In particular, we show that if  $n > 3k$ , the conjecture is valid. We checked the conjecture with our table for  $n < 78$  and it had no inconsistency. Finally, we show that for every fixed  $k$ ,  $\alpha(P(n, k))$  can be computed using an algorithm with running time  $O(n)$ .

**Keywords:** Generalized Petersen Graphs, Independent Set, Tree Decomposition

## 1 Introduction and preliminaries

In a graph  $G = (V, E)$ , an **independent set**  $I(G)$  is a subset of the vertices of  $G$  such that no two vertices in  $I(G)$  are adjacent. The **independence number**  $\alpha(G)$  is the cardinality of a largest set of independent vertices and an independent set of size  $\alpha(G)$  is called an  $\alpha$ -set. The maximum independent set problem is to find an independent set with the largest

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number of vertices in a given graph. It is well-known that this problem is NP-hard [10]. Therefore, many attempts are made to find upper and lower bounds, or exact values of  $\alpha(G)$  for special classes of graphs. This paper is aimed toward studying this problem for the generalized Petersen graphs.

For each  $n$  and  $k$  ( $n > 2k$ ), a generalized Petersen graph  $P(n, k)$ , is defined by vertex set  $\{u_i, v_i\}$  and edge set  $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k}\}$ ; where  $i = 1, 2, \dots, n$  and subscripts are reduced modulo  $n$ . An induced subgraph on  $v$ -vertices is called the **inner subgraph**, and an induced subgraph on  $u$ -vertices is called the **outer cycle**.

In addition, we call two vertices  $u_i$  and  $v_i$  as twin of each other and the edge between them as a spoke.

In [8], Coxeter introduced this class of graphs. Later Watkins [14] called these graphs “generalized Petersen graphs”,  $P(n, k)$ , and conjectured that they admit a Tait coloring, except  $P(5, 2)$ . This conjecture later was proved in [7]. Since 1969 this class of graphs has been studied widely. Recently vertex domination and minimum vertex cover of  $P(n, k)$  have been studied. For more details see for instance [2], [3], [4] and [11].

A set  $Q$  of vertices of a graph  $G = (V, E)$  is called a **vertex cover** of  $G$  if every edge of  $G$  has at least one endpoint in  $Q$ . A vertex cover of a graph  $G$  with the minimum possible cardinality is called a **minimum vertex cover** of  $G$  and its size is denoted by  $\beta(G)$ . In [2] and [4]  $\beta(P(n, k))$ , has been studied. Since for every simple graph  $G$ ,  $\alpha(G) + \beta(G) = |V(G)|$  [15], their results imply the following results for  $\alpha(P(n, k))$ , and  $n > 2k$ :

$$\text{I) } \alpha(P(n, 1)) = \begin{cases} n & n \text{ is even} \\ n-1 & n \text{ is odd.} \end{cases}$$

$$\text{II) For all } n > 4, \alpha(P(n, 2)) = \lfloor \frac{4n}{5} \rfloor.$$

$$\text{III) For all } n > 6, \alpha(P(n, 3)) = \begin{cases} n & n \text{ is even} \\ n-2 & n \text{ is odd.} \end{cases}$$

$$\text{IV) If both } n \text{ and } k \text{ are odd, then } \alpha(P(n, k)) \geq n - \frac{k+1}{2}.$$

$$\text{Also, if } k \mid n, \text{ then } \alpha(P(n, k)) = n - \frac{k+1}{2}.$$

$$\text{V) } \alpha(P(n, k)) = n \text{ if and only if } n \text{ is even and } k \text{ is odd.}$$

$$\text{VI) For all even } k, \text{ we have}$$

$$- \text{ If } k-1 \mid n \text{ then } \alpha(P(n, k)) \geq n - \frac{n}{k-1}.$$

$$- \text{ If } k-1 \nmid n \text{ then } \alpha(P(n, k)) > n - \frac{n}{k-1} - 2k.$$

$$\text{VII) For all odd } n, \text{ we have } \alpha(P(n, k)) \leq n - \frac{d+1}{2}, \text{ where } d = \gcd(n, k).$$

Recently, Fox et al. proved the following results in [9]:

VIII) For all  $n > 10$ ,  $\alpha(P(n, 5)) = \begin{cases} n & n \text{ is even} \\ n-3 & n \text{ is odd} \end{cases}$

IX) For any integer  $k \geq 1$ , we have that  $\alpha(P(3k, k)) = \lceil \frac{5k-2}{2} \rceil$ .

X) If  $n, k$  are integers with  $n$  odd and  $k$  even, then  $\alpha(P(n, k)) \geq \frac{n-1}{2} + \lfloor \frac{\lceil \frac{n}{k} \rceil + 1}{2} \rfloor \cdot \lfloor \frac{n}{2d \lceil \frac{n}{k} \rceil} \rfloor + \frac{d-1}{2} \cdot \lfloor \frac{1}{2} \left( \frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil} \right) \rfloor$ , where  $d = \gcd(n, k)$ .

XI) If  $n, k$  are even, then  $\alpha(P(n, k)) \geq \frac{n}{2} + \frac{d}{2} \lfloor \frac{n}{2d} \rfloor$ , where  $d = \gcd(n, k)$ .

Notice that the problem of finding the size of a maximum independent set in the graph  $P(n, k)$  is trivial for even  $n$  and odd  $k$ , since  $P(n, k)$  is a bipartite graph. For odd  $n$  and  $k$ ,  $P(n, k)$  is not bipartite but we can remove a set of  $k+1$  edges from  $P(n, k)$  to obtain a bipartite graph. Thus in this case we have  $n - (k+1) \leq \alpha(P(n, k)) \leq n$ . So, for odd  $k$ , we have upper and lower bounds for  $\alpha(P(n, k))$  that are at most  $k+1$  away from  $n$ . In contrast, for even  $k$ ,  $P(n, k)$  has a lot of odd cycles. In fact, the number of odd cycles in  $P(n, k)$  is at least as large as  $O(n)$ . This observation shows that for even  $k$ , the graph  $P(n, k)$  is far from being a bipartite graph and as we see in continuation, we need more complicated arguments for finding lower and upper bounds for  $\alpha(P(n, k))$  compared to the case that  $k$  is an odd number.

This paper is organized as follows. In Section 2, we provide an upper bound for  $\alpha(P(n, k))$  for even  $k > 2$ . In Section 3, we present some lower bounds when  $k$  is even. In both Section 2 and Section 3 we compare our bounds with the previously existing bounds. Some exact values for  $\alpha(P(n, k))$  are given in Section 4 by applying results presented in Sections 2 and Section 3. Finally, in Section 5 we prove Behsaz-Hatami-Mahmoodian's conjecture for some cases by using known lower bounds. We checked the conjecture with our Table for  $n < 78$ , and it had no inconsistency.

## 2 Upper bound

In this section we present an upper bound for  $\alpha(P(n, k))$  when  $k > 2$  is even and we will show that the presented upper bound is equal to  $\alpha(P(n, k))$  in some cases. Our upper bound is better than the upper bound given by Behsaz et.al. in [2].

For  $t = 1, 2, \dots, n$ , we call the set  $\{u_t, u_{t+1}, \dots, u_{t+2k-1}, v_t, v_{t+1}, \dots, v_{t+2k-1}\}$  a  $2k$ -segment and we denote it by  $I_t$ . Let  $G[I_t]$  be the subgraph of  $P(n, k)$  induced by  $I_t$ .

Let  $\mathcal{S}$  be the set of all maximum independent sets of  $P(n, k)$ . For every  $S \in \mathcal{S}$  we denote by  $f(S)$  the number of  $2k$ -segments  $I_t$  for which  $|I_t \cap S| = 2k$  ( $1 \leq t \leq n$ ).

Define  $\mathcal{S}_{\min} = \{S \in \mathcal{S} \mid \forall S' \in \mathcal{S}, f(S) \leq f(S')\}$ . Since  $\mathcal{S}$  is nonempty,  $\mathcal{S}_{\min}$  is also nonempty. Let  $S_0 \in \mathcal{S}_{\min}$ .

**Proposition 1** *For any  $S \in \mathcal{S}$ ,  $S \in \mathcal{S}_{\min}$  if and only if  $f(S) = f(S_0)$ .*

**Definition 1** *For any  $S \in \mathcal{S}$ , we say  $I_t$  is of Type 1 with respect to  $S$  if  $|I_t \cap S| = 2k$ , of Type 2 with respect to  $S$  if  $|I_t \cap S| = 2k - 1$ , and of Type 3 with respect to  $S$  if  $|I_t \cap S| \leq 2k - 2$ .*

*Let  $T_i(S) = \{I_t \mid I_t \text{ is of Type } i \text{ with respect to } S\}$ , for  $i = 1, 2, 3$ . For a given  $I_t \in T_2(S)$ , we say  $I_t$  is of Special type 2 with respect to  $S$ , if  $u_t \notin S$  and  $\{u_t\} \cup (I_t \cap S)$  becomes an independent set for  $G[I_t]$ .*

Since  $G[I_t]$  has a perfect matching of spokes  $\{u_t v_t, u_{t+1} v_{t+1}, \dots, u_{2k+t-1} v_{2k+t-1}\}$ ,  $|I_t \cap S| \leq 2k$ . So every  $I_t$  is one of the above Types. Note that  $f(S) = |T_1(S)|$ .

**Lemma 1** *If  $k$  is an even number then  $\alpha(G[I_t]) = 2k$  and  $G[I_t]$  has a unique  $\alpha$ -set shown in Figure 1.*

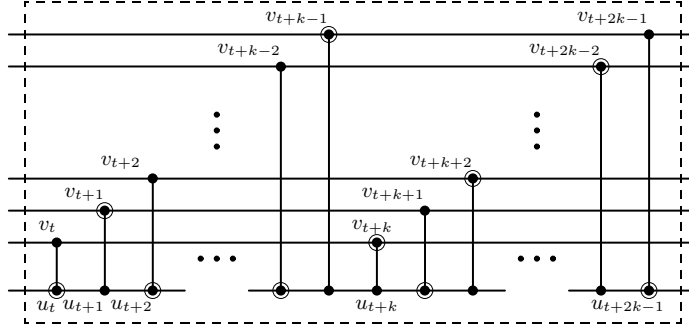


Figure 1:  $I_t$  as a Type 1 segment.

**Proof**  $G[I_t]$  has a perfect matching  $\{u_i v_i \mid t \leq i \leq t + 2k - 1\}$ . So  $\alpha(G[I_t]) \leq \frac{|V(G[I_t])|}{2} = 2k$ . On the other hand, Figure 1 is an example of an independent set of  $G[I_t]$  of size  $2k$ . So  $\alpha(G[I_t]) = 2k$ .

To show that  $G[I_t]$  has a unique  $\alpha$ -set, let  $S$  be an  $\alpha$ -set of  $(G[I_t])$ . Since  $\alpha(G[I_t]) = 2k$ ,  $|S| = 2k$  and  $S$  must contain precisely one vertex from each edge  $\{u_i, v_i\}$  where  $t \leq i \leq t + 2k - 1$ . Notice that the set of  $u$ -vertices of  $(G[I_t])$  induces a path of length  $2k$ . Therefore  $|S \cap \{u_i \mid t \leq i \leq t + 2k - 1\}| \leq k$ . The set of  $v$ -vertices of  $(G[I_t])$  induces a matching of size  $k$ . This means that  $|S \cap \{v_i \mid t \leq i \leq t + 2k - 1\}| \leq k$ . These two observations show that

any  $\alpha$ -set  $S$  of  $G[I_t]$  has  $k$  vertices from  $u$ -vertices and  $k$  other vertices from  $v$ -vertices of  $V(G[I_t])$ . Moreover, every such  $S$  contains precisely one vertex from each edge  $u_i v_i$  where  $t \leq i \leq t + 2k - 1$  and  $v_i v_{i+k}$  where  $t \leq i \leq t + k - 1$ . Now, consider two cases:

Case 1:  $v_t \in S$ .

In this case,  $u_t$  and  $v_{t+k}$  are forced not to be in  $S$ . So  $u_{t+k}$  is forced to be in  $S$ . Then  $u_{t+k-1}$  and  $u_{t+k+1}$  are forced not to be in  $S$  and this forces  $v_{t+k-1}$  and  $v_{t+k+1}$  to be in  $S$ . Since  $v_{t+k+1}$  is in  $S$ ,  $v_{t+1} \notin S$ . Therefore  $u_{t+1} \in S$ , so  $u_{t+2} \notin S$  and thus  $v_{t+2} \in S$ . So, we showed that if  $v_t \in S$  then  $v_{t+2} \in S$  too. Now, if we repeat the same argument for  $v_{t+2}$  instead of  $v_t$ , we can deduce that  $v_{t+4} \in S$  and by a simple induction, it follows that  $v_{t+2l} \in S$  for any  $0 \leq l \leq \frac{k}{2} - 1$ . Particularly,  $v_{t+k-2} \in S$ . Therefore  $v_{t+2k-2} \notin S$ . This shows that  $u_{t+2k-2} \in S$ . Hence  $u_{t+2k-1} \notin S$  and  $v_{t+2k-1} \in S$ . So  $v_{t+k-1} \notin S$ . But we already showed that  $v_{t+k-1}$  is forced to be in  $S$ . This contradiction shows that there is no Type 1  $I_t$  for which  $v_t \in S$ .

Case 2:  $u_t \in S$ .

In this case, similar to the argument in Case 1, each vertex is either forced to be in  $S$  or it is forced not to be in  $S$ . So, there is a unique pattern for  $S \cap I_t$  when  $I_t \in T_1(S)$ . Since the pattern shown in Figure 1 is an instance of an independent set of size  $2k$  for  $G[I_t]$ , it is the unique pattern for such an independent set. ■

Lemma 1 guarantees that there is a unique pattern for  $I_t \cap S$ , if  $I_t$  is of Special type 2 with respect to  $S$ .

**Lemma 2** *For every  $S \in \mathcal{S}$ , if  $I_t \in T_1(S)$  then  $u_{2k+t}, v_{2k+t}, u_{t-1}$ , and  $v_{t-1} \notin S$ . Also, if  $I_t$  is a Special type 2 segment with respect to  $S$  then  $u_{2k+t}$  and  $v_{2k+t} \notin S$ .*

**Proof** If  $I_t \in T_1(S)$  then  $|I_t \cap S| = 2k$ . So by Lemma 1, there is a unique pattern for  $I_t \cap S$ . Based on this pattern,  $u_{2k+t-1}$  and  $v_{k+t} \in S$ . Therefore  $u_{2k+t}$  and  $v_{2k+t} \notin S$ , since  $S$  is an independent set of vertices of  $P(n, k)$ . A similar argument shows that  $u_{t-1}$  and  $v_{t-1} \notin S$ . The proof of the second part of the lemma is similar. ■

**Corollary 1** *If  $I_t \in T_1(S)$  then  $I_{t+1}, I_{t+2}, \dots, I_{t+2k} \notin T_1(S)$ .*

**Proof** Notice that if  $I_r \in T_1(S)$  then for any edge  $u_i v_i \in E(G[I_r])$  either  $u_i \in S$  or  $v_i \in S$ . Since  $I_t \in T_1(S)$ , Lemma 2 implies that  $u_{2k+t}$  and  $v_{2k+t} \notin S$ . On the other hand,  $u_{2k+t} v_{2k+t} \in E(G[I_{t+i}])$  for  $i = 1, 2, \dots, 2k$ . Thus  $I_{t+1}, I_{t+2}, \dots, I_{t+2k} \notin T_1(S)$ . ■

**Theorem 1**  $\alpha(P(n, k)) \leq \lfloor \frac{(2k-1)n}{2k} \rfloor$  for any even number  $k > 2$  and any integer  $n > 2k$ .

**Proof** Let  $S_0 \in \mathcal{S}_{\min}$ . We consider two cases.

Case 1:  $f(S_0) = 0$ .

In this case  $T_1(S_0) = \emptyset$ . So  $|I_t \cap S_0| \leq 2k - 1$  for any  $1 \leq t \leq n$ . If we add all of these  $n$  inequalities we get:

$$\sum_{t=1}^n |I_t \cap S_0| \leq (2k - 1)n. \quad (1)$$

On the other hand  $\sum_{t=1}^n |I_t \cap S_0| = 2k|S_0|$ , since every element of  $S_0$  is contained in precisely  $2k$  of the sets  $I_t$ . Thus:

$$2k|S_0| \leq (2k - 1)n \implies \alpha(P(n, k)) = |S_0| \leq \frac{2k - 1}{2k}n.$$

Case 2:  $f(S_0) > 0$ .

In this case  $T_1(S_0) \neq \emptyset$ . Similar to the inequality 1 we have:

$$2k|S_0| = \sum_{t=1}^n |I_t \cap S_0| \leq (2k - 1)n + |T_1(S_0)| - |T_3(S_0)|.$$

So, to prove the theorem, it suffices to show that there exists  $S_0 \in \mathcal{S}_{\min}$  such that  $|T_1(S_0)| \leq |T_3(S_0)|$ .

If we can show that for any  $I_r \in T_1(S_0)$ , there exists an  $I_{r'} \in T_3(S_0)$  so that  $I_{r+1}, I_{r+2}, \dots, I_{r'} \notin T_1(S_0)$ , then it follows that  $|T_1(S_0)| \leq |T_3(S_0)|$ .

On the contrary, suppose that there exists  $I_t \in T_1(S_0)$  in such a way that in the sequence  $I_{t+1}, I_{t+2}, \dots$  before we see an element of  $T_3(S_0)$ , we see an element of  $T_1(S_0)$ . Without loss of generality we can assume that  $t = 1$ . By Lemma 1,  $(I_1 \cap S_0)$  is of the form depicted in Figure 2.

Since  $I_1 \in T_1(S_0)$ , by Corollary 1,  $I_2, I_3, \dots, I_{2k+1} \notin T_1(S_0)$ . Based on

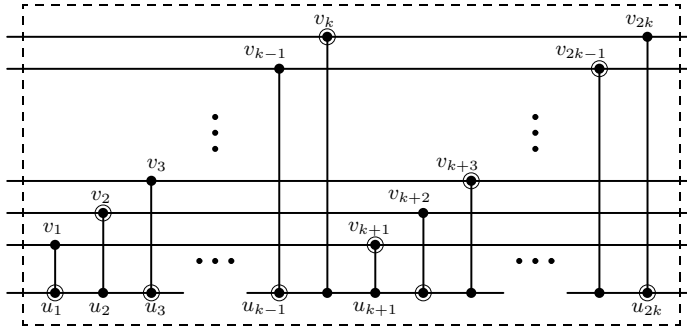


Figure 2:  $I_1$  as a Type 1 segment.

our assumption,  $I_2, I_3, \dots, I_{2k+1} \in T_2(S_0)$ . In particular,  $I_{2k+1} \in T_2(S_0)$ . Since  $I_1 \in T_1(S_0)$ , by Lemma 2 we have  $u_{2k+1}, v_{2k+1} \notin S_0$ . On the other hand, we know that  $S_0$  must have one vertex from each edge  $u_i v_i$  where  $2k+2 \leq i \leq 4k$ . Since  $2k+2 \leq 2k+3 \leq 4k$ , either  $u_{2k+3}$  or  $v_{2k+3} \in S_0$ . But notice that  $v_{2k+3}$  is adjacent to  $v_{k+3}$  which is in  $S_0$ , for  $k > 2$ . Thus,  $v_{2k+3} \notin S_0$  and  $u_{2k+3}$  must be in  $S_0$ . This means that  $u_{2k+2} \notin S_0$ . Now, define  $S_1 := (S_0 \setminus \{u_{2k}\}) \cup \{u_{2k+1}\}$ . One can easily see that  $S_1 \in \mathcal{S}$ . Based on the choice of  $S_0 \in \mathcal{S}_{\min}$ ,  $f(S_0) \leq f(S_1)$ . Therefore, there must be an index  $2 \leq r \leq n$  so that  $I_r \in T_1(S_1) \setminus T_1(S_0)$ . Since  $S_0$  and  $S_1$  agree on every element except  $u_{2k}$  and  $u_{2k+1}$ , the only candidate for  $r$  is  $r = 2k+1$ . So  $I_{2k+1} \in T_1(S_1)$  and  $I_{2k+1} \notin T_1(S_0)$ . Moreover  $I_1 \in T_1(S_0)$  and  $I_1 \notin T_1(S_1)$ . Thus  $f(S_0) = f(S_1)$ . By Proposition 1,  $S_1 \in \mathcal{S}_{\min}$ . Notice that if any of  $I_{2k+2}, I_{2k+3}, \dots, I_n$  are of Type  $i$  with respect to  $S_1$ , they are of Type  $i$  with respect to  $S_0$ , as well. So, in the sequence  $I_{2k+2}, I_{2k+3}, \dots, I_n$ , any Type 3 segment with respect to  $S_1$  appears after an element of Type 1 with respect to  $S_1$ . Since  $I_{2k+1} \in T_1(S_1)$  by Corollary 1,  $I_{2k+2}, I_{2k+3}, \dots, I_{4k+1} \notin T_1(S_1)$ . Then from our assumption  $I_{2k+2}, I_{2k+3}, \dots, I_{4k+1} \in T_2(S_1)$ .

This means that the same argument can be applied to  $S_1$  and if we define  $S_2 := (S_1 \setminus \{u_{4k}\}) \cup \{u_{4k+1}\}$ , then  $S_2 \in \mathcal{S}_{\min}$ . If we consecutively repeat this argument for  $S_1, S_2, S_3, \dots, S_m$  where  $m = \lfloor \frac{n}{2k} \rfloor$  and  $S_i := (S_{i-1} \setminus \{u_{2ik}\}) \cup \{u_{2ki+1}\}$ , then we observe that  $S_i \in \mathcal{S}_{\min}$  and  $I_{2ki+1} \in T_1(S_i)$  for  $i = 1, 2, \dots, m$ , and none of  $I_2, I_3, \dots, I_{2k(m+1)}, I_{2k(m+1)+1}$  are of Type 1 with respect to  $S_0$ . Also,  $I_{2k(i+1)+1}$  for  $i = 0, 1, \dots, m$  are of Special type 2 with respect to  $S_i$ . Since  $S_i$  and  $S_0$  agree on the  $I_{2ki+2}, I_{2ki+3}, \dots, I_n$ , then  $I_{2k(i+1)+1}$  for  $i = 0, 1, \dots, m$  are of Special type 2 with respect to  $S_0$ .

In other words, if  $I_1$  belongs to  $T_1(S_0)$  and the next element of  $T_1(S_0)$  appears before the first element of  $T_3(S_0)$  in the sequence  $I_2, I_3, I_4, \dots$ , then all of  $I_{2k+1}, I_{4k+1}, \dots, I_{2km+1}, I_{2k(m+1)+1}$  are Special type 2 with respect to  $S_0$ . In particular,  $I_{2km+1}$  is of Special type 2 with respect to  $S_0$ .

As  $I_{2km+1} \in T_1(S_m)$ , by Lemma 2,  $u_{2k(m+1)+1}, v_{2k(m+1)+1} \notin S_m$  and since  $S_m$  and  $S_0$  agree on the  $I_{2km+2}, I_{2km+3}, \dots, I_n$  we conclude that  $u_{2k(m+1)+1}, v_{2k(m+1)+1} \notin S_0$ .

Now consider three cases:

- $2k(m+1) + 1 \equiv 1 \pmod{n}$ :  
Since  $I_{2km+1}$  is of Special type 2 with respect to  $S_0$ , by Lemma 2, we have  $u_{2km+1+2k} = u_1 \notin S_0$ . This is a contradiction as we assumed  $I_1 \in T_1(S_0)$  and therefore  $u_1 \in S_0$ .
- $2k(m+1) + 1 \not\equiv 0, 1 \pmod{n}$ :  
Since  $I_1 \in T_1(S_0)$ , by Lemma 2,  $u_n, v_n \notin S_0$ . Also we know that

$u_{2k(m+1)+1}, v_{2k(m+1)+1} \notin S_0$ . Thus,  $I_{2k(m+1)+1}$  is of Type 3 with respect to  $S_0$  and none of  $I_2, I_3, \dots, I_{2k(m+1)}$  are of Type 1 with respect to  $S_0$  which is a contradiction.

- $2k(m+1) + 1 \equiv 0 \pmod{n}$ :  
 $I_1$  is of Type 1 with respect to  $S_0$  and for every  $1 \leq i \leq m$ ,  $I_{2k(i+1)+1}$  is of Special type 2 with respect to  $S_0$ . In particular,  $I_{2km+1}$  is of Special type 2 with respect to  $S_0$ , and therefore  $v_{2km+k+2} = v_{2k(m+1)-k+3} \in S_0$ , (See Figure 3). On the other hand,  $v_2 \in S_0$  as  $I_1 \in T_1(S_0)$ , and since  $n = 2k(m+1) + 1$ ,  $v_2$  is adjacent to  $v_{2k(m+1)-k+3}$ . This is a contradiction.

So in all the cases, we get a contradiction which means, after any Type 1 segment  $I_r$ , a Type 3 segment  $I_{r'}$  will appear before we see another Type 1 segment. This means that  $|T_1(S_0)| \leq |T_3(S_0)|$  and the theorem follows, as we argued earlier.  $\blacksquare$

### 3 Lower bounds

In this section we introduce some lower bounds for  $\alpha(P(n, k))$  where  $k$  is even and  $k > 2$ .

Here we explain a construction for an independent set in  $P(n, k)$  for even numbers  $n$  and  $k$ . It happens that for every even  $n < 78$ , our lower bound is equal to the actual value, using a computer program for finding the independence number in  $P(n, k)$ .

**Theorem 2** *If  $n$  and  $k$  are even and  $k > 2$  then:*

$$\alpha(P(n, k)) \geq (2k-1) \lfloor \frac{n}{2k} \rfloor + \begin{cases} \frac{r}{2} & \text{if } r \leq k, \\ \frac{3r}{2} - k - 1 & \text{if } r > k. \end{cases}$$

where  $r$  is the remainder of  $n$  modulo  $2k$ .

**Proof** We partition the vertices of  $P(n, k)$  into  $\lfloor \frac{n}{2k} \rfloor$   $2k$ -segments and one  $r$ -segment. Since  $n$  and  $k$  are even numbers,  $r$  is also an even number and it is straightforward to see that if we choose a subset of the form shown in Figure 3, from each  $2k$ -segment they form an independent set  $S_0$  of size  $(2k-1) \lfloor \frac{n}{2k} \rfloor$ .

Then we try to extend this independent set by adding more vertices from the remaining  $r$ -segment. Without loss of generality, we may assume that the  $r$ -segment consists of the vertices  $\{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_r\}$ . consider two cases:



- $r \leq k$  :

In this case the set  $S_0 \cup \{u_2, u_4, \dots, u_{r-2}, u_r\}$  is an independent set of size  $(2k-1)\lfloor \frac{n}{2k} \rfloor + \frac{r}{2}$ .

- $r > k$  :

In this case the set:

$S_0 \cup \{u_3, u_5, \dots, u_{r-k-3}, u_{r-k-1}\} \cup \{v_2, v_4, \dots, v_{r-k-2}, v_{r-k}\} \cup \{u_{r-k+1}, u_{r-k+3}, \dots, u_{k-3}, u_{k-1}\} \cup \{u_{k+2}, u_{k+4}, \dots, u_{r-2}, u_r\} \cup \{v_{k+1}, v_{k+3}, \dots, v_{r-3}, v_{r-1}\}$  is an independent set of size  $(2k-1)\lfloor \frac{n}{2k} \rfloor + \frac{r-k-2}{2} + \frac{r-k}{2} + \frac{2k-r}{2} + \frac{r-k}{2} + \frac{r-k}{2} = (2k-1)\lfloor \frac{n}{2k} \rfloor + \frac{3r}{2} - k - 1$ .

■

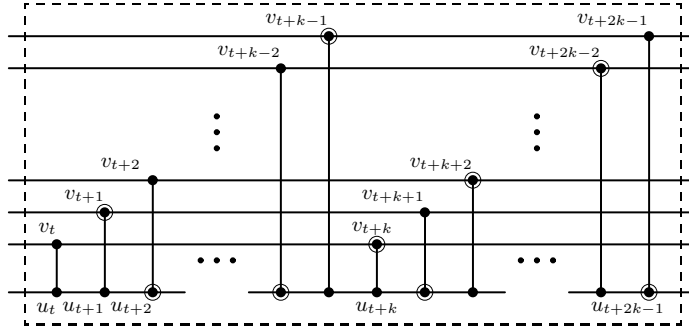


Figure 3:  $I_t$  as a Special type 2 segment.

In the next theorem we establish a lower bound for  $\alpha(P(n, k))$  for odd  $n$  and even  $k$ .

**Theorem 3** *If  $n$  is odd and  $k > 2$  is even then we have:*

$$\alpha(P(n, k)) \geq (2k-1)\lfloor \frac{n}{2k} \rfloor + \begin{cases} -\frac{k}{2} + 2 & \text{if } r = 1, \\ \frac{3r-k-1}{2} & \text{if } 1 < r < k, \\ \frac{k}{2} + \frac{r-1}{2} & \text{if } k < r < 2k. \end{cases}$$

where  $r$  is the remainder of  $n$  modulo  $2k$ .

**Proof** We construct an independent set for the graph  $P(n, k)$ . Similar to the proof of Theorem 2, first we partition the vertices of the graph into  $\lfloor \frac{n}{2k} \rfloor$   $2k$ -segments and a remaining segment of size  $r$ . Without loss of generality,

we can assume that the last  $2k$ -segment starts from the first spoke and the remaining segment starts from the  $(2k+1)$ -st spoke and finishes at the  $(2k+r)$ -th spoke. We also label the  $2k$ -segments with indices  $1, 2, \dots, \lfloor \frac{n}{2k} \rfloor$ .

From each of  $2k$ -segments  $1, 2, \dots, \lfloor \frac{n}{2k} \rfloor - 1$ , we choose  $2k-1$  vertices as shown in Figure 3. We also choose the following vertices from the last  $2k$ -segment and the remaining  $r$ -segment:

- $r = 1$ :  
 $\{u_2, u_4, \dots, u_{k-2}, u_k\} \cup \{u_{k+3}, u_{k+5}, \dots, u_{2k-1}, u_{2k+1}\} \cup \{v_{k+1}\} \cup \{v_{k+2}, v_{k+4}, \dots, v_{2k-2}, v_{2k}\}.$
- $1 < r < k$  :  
 $\{u_3, u_5, \dots, u_{k-3}, u_{k-1}\} \cup \{u_{k+2}, u_{k+4}, \dots, u_{2k-2}, u_{2k}\} \cup \{u_{2k+3}, u_{2k+5}, \dots, u_{2k+r-2}, u_{2k+r}\} \cup \{v_2, v_4, \dots, v_{k-2}, v_k\} \cup \{v_{k+1}, v_{k+3}, \dots, v_{k+r-2}, v_{k+r}\} \cup \{v_{2k+2}, v_{2k+4}, \dots, v_{2k+r-3}, v_{2k+r-1}\}.$
- $k < r < 2k$  :  
 $\{u_3, u_5, \dots, u_{k-3}, u_{k-1}\} \cup \{u_{k+2}, u_{k+4}, \dots, u_{2k-2}, u_{2k}\} \cup \{u_{2k+3}, u_{2k+5}, \dots, u_{2k+r-2}, u_{2k+r}\} \cup \{v_2, v_4, \dots, v_{k-2}, v_k\} \cup \{v_{k+1}, v_{k+3}, \dots, v_{2k-3}, v_{2k-1}\} \cup \{v_{2k+2}, v_{2k+4}, \dots, v_{3k-2}, v_{3k}\}.$

One can easily check that in each case, the given set is an independent set of size specified in the theorem. ■

Notice that the upper bound given in Theorem 1 and the lower bound in Theorem 2, and Theorem 3 are very close to each other for every fixed even  $k > 2$ . More precisely, we have the following corollary:

**Corollary 2** *If  $k > 2$  is an even number then  $\alpha(P(n, k)) = \frac{(2k-1)}{2k}n + O(k)$ .*

Notice that our lower bounds are considerably better than the lower bounds obtained in [2] and [9].

## 4 Some exact values

In this section, we will find the exact value of  $\alpha(P(n, k))$  for some pairs of  $n, k$ .

**Proposition 2** *If  $n > 8$ , then:*

$$\alpha(P(n, 4)) = \begin{cases} \frac{7n}{8} & \text{if } n \equiv 0 \pmod{8}, \\ \frac{7}{8}(n-1) & \text{if } n \equiv 1 \pmod{8}, \\ \frac{7}{8}(n-2) + 1 & \text{if } n \equiv 2 \pmod{8}, \\ \frac{7}{8}(n-3) + 2 & \text{if } n \equiv 3 \pmod{8}, \\ \frac{7}{8}(n-5) + 4 & \text{if } n \equiv 5 \pmod{8}. \end{cases}$$

$$\alpha(P(n, 4)) \geq \begin{cases} \frac{7}{8}(n-4) + 2 & \text{if } n \equiv 4 \pmod{8}, \\ \frac{7}{8}(n-6) + 4 & \text{if } n \equiv 6 \pmod{8}, \\ \frac{7}{8}(n-7) + 5 & \text{if } n \equiv 7 \pmod{8}. \end{cases}$$

**Proof** This result is straight consequence of Theorems 1, 2, and 3. ■

Notice that for  $k = 4$  and  $n \equiv 4, 6$  or  $7 \pmod{8}$ , the upper bound and lower bound differ by 1. In fact, for  $n < 700$  the exact of  $\alpha(P(n, k))$  is the same as our lower bound as we checked by computer.

**Proposition 3** *If  $k > 2$  is an even number and  $n \equiv 0, 2, k-1$  or  $k+1 \pmod{2k}$  then  $\alpha(P(n, k)) = \lfloor \frac{(2k-1)n}{2k} \rfloor$ .*

**Proof** This assertion is trivial consequence of Theorems 1, 2 and 3. In fact the upper bound and lower bounds we have for  $\alpha(P(n, k))$  are identical in these cases. ■

## 5 Behsaz-Hatami-Mahmoodian's conjecture

**Conjecture** ([2]). For all  $n, k$  we have  $\beta(P(n, k)) \leq n + \lceil \frac{n}{5} \rceil$ .

Notice that, since  $\alpha(G) + \beta(G) = |V(G)|$ , this conjecture is equivalent to  $\alpha(P(n, k)) \geq \lfloor \frac{4n}{5} \rfloor$ .

**Theorem 4** *The above conjecture is valid in the following cases:*

- a)  $k = 1, 2, 3, 4, 5$ .
- b)  $n$  is even and  $k$  is odd.
- c)  $n, k$  are odd and  $n \geq \frac{5(k+1)}{2}$ .
- d)  $n, k$  are even.

e)  $n$  is odd,  $k$  is even and  $n > 3k$ .

**Proof** a) This case is a straight consequence of (I), (II), (III), Proposition 2 and (VIII).

b) In this case  $P(n, k)$  is a bipartite graph and  $\alpha(P(n, k)) = n$ .

c)  $\alpha(P(n, k)) \geq n - \frac{k+1}{2}$  ([2]). For  $n \geq \frac{5(k+1)}{2}$  this lower bound is greater than  $\lfloor \frac{4n}{5} \rfloor$ .

d) Let  $n = 2kq + r$  where  $q \geq 1$  and  $0 \leq r < 2k$ . We consider the following subcases:

- If  $r \leq k$  and  $q = 1$  then by Theorem 2,  $\alpha(P(n, k)) \geq 2k - 1 + \frac{r}{2} \geq \frac{4}{5}(2k + r)$  for any  $k \geq 10$ . For  $k < 10$  conjecture holds based on the information provided in Table 1.
- If  $r \leq k$  and  $q > 1$  then by Theorem 2,  $\alpha(P(n, k)) \geq (2k - 1)q + \frac{r}{2} \geq \frac{4}{5}(2kq + r)$  for any  $k \geq 4$ . For  $k < 4$ , conjecture follows from (a).
- If  $r > k$  and  $q = 1$  then by Theorem 2,  $\alpha(P(n, k)) \geq 2k - 1 + \frac{3r}{2} - k - 1 \geq \frac{4}{5}(2k + r)$  for any  $k \geq 20$ . For  $k < 20$  conjecture holds based on the information provided in Table 1.
- If  $r > k$  and  $q > 1$  then by Theorem 2,  $\alpha(P(n, k)) \geq (2k - 1)q + \frac{3r}{2} - k - 1 \geq \frac{4}{5}(2kq + r)$  for any  $k \geq 6$ . For  $k < 6$ , conjecture follows from (a).

e) Similar to the previous part, let  $n = 2kq + r$  where  $q \geq 1$  and  $0 \leq r < 2k$ . We consider the following subcases:

- If  $r = 1$  and  $q = 1$  then  $P(n, k) = P(2k + 1, k)$  which is isomorphic to  $P(2k + 1, 2)$ . (For more information about isomorphic generalized Petersen graphs see [13]).
- If  $r = 1$  and  $q \geq 2$  then by Theorem 3,  $\alpha(P(n, k)) \geq (2k - 1)q - \frac{k}{2} + 2 \geq \frac{4}{5}(2kq + 1)$  for every  $k > 2$ .
- If  $1 < r < k$  then  $q$  has to be larger than 1. In fact if  $1 < r < k$  and  $q = 1$  then  $n = 2kq + r < 3k$ . For  $1 < r < k$  and  $q \geq 2$  then by Theorem 3,  $\alpha(P(n, k)) \geq (2k - 1)q + \frac{3r - k - 1}{2} \geq \frac{4}{5}(2kq + r)$  for every  $k \geq 2$ . (Note that since  $n$  is odd and  $k$  is even,  $r > 1$  implies that  $r \geq 3$ ).
- If  $k < r < 2k$  then by Theorem 3,  $\alpha(P(n, k)) \geq (2k - 1)q + \frac{k}{2} + \frac{r - 1}{2} \geq \frac{4}{5}(2kq + r)$  for  $k \geq 5$ . For  $k < 5$  then the assertion is concluded from part (a).

■

**Corollary 3** *If  $n > 3k$  then  $\alpha(P(n, k)) \geq \lfloor \frac{4n}{5} \rfloor$ , and Behsaz-Hatami-Mahmoodian's conjecture holds.*

## 6 Polynomial algorithm for $\alpha(P(n, k))$

In this section we will prove that the independence number of generalized Petersen graphs with fixed  $k$  can be found in linear time,  $O(n)$ . This result is a special case of a deep theorem stating that the problem of finding the independence number of graphs with bounded treewidth can be solved in linear time of the number of vertices of the graph.

In the continuation we will show that for every fixed  $k$  and any integer  $n > 2k$  the treewidth of  $P(n, k)$  is bounded. First, we need to formally define the concepts of tree decomposition and treewidth of a graph.

**Definition 2** *Let  $G = (V, E)$  be a graph. A tree decomposition of  $G$  is a pair  $(X, T)$ , where  $X = \{X_1, X_2, \dots, X_n\}$  is a family of subsets of  $V$ , and  $T$  is a tree whose nodes are the subsets  $X_j$ , satisfying the following properties:*

- 1) *The union of all sets  $X_j$  equals  $V$ .*
- 2) *For every edge  $(v, w)$  in the graph, there is a subset  $X_j$  that contains both  $v$  and  $w$ .*
- 3) *If  $X_j$  is on the path from  $X_i$  to  $X_l$  in  $T$  then  $X_i \cap X_l \subseteq X_j$ . In other words, for all vertices  $v \in V$ , all nodes  $X_i$  which contain  $v$  induce a connected subtree of  $T$ .*

*The width of  $(X, T)$  is defined to be the size of the largest  $X_i$  minus one. The treewidth,  $tw(G)$ , of the graph  $G$  is defined to be the minimum width of all its tree decompositions. The treewidth will be taken as a measure of how much a graph resembles a tree.*

**Theorem A ([5]).** *The problem of finding a maximum independent set of a graph  $G$  with bounded treewidth,  $tw(G) \leq l$  can be solved in  $O(2^{3l}n)$  by dynamic programming techniques, where  $n$  is the number of vertices of graph.*

For more details see for instance [1], [5], [6], and [12].

**Theorem 5** *For any fixed  $k$ , the problem of finding independence number of the graphs  $P(n, k)$  can be solved by an algorithm with running time  $O(n)$ .*

**Proof** By Theorem A, we only need to show that for a given number  $k$ , the treewidth of  $P(n, k)$  is bounded. Consider the following tree decomposition of  $P(n, k)$  of width  $4k + 3$ . Without loss of generality we can only consider the case where  $n > 2k + 1$ . Let  $T$  be the path of order  $n - 2k - 1$  and define  $X_1, X_2, \dots, X_{n-2k-1}$  as follows:

$$X_1 = \{u_1, v_1, u_2, v_2, \dots, u_{k+1}, v_{k+1}, u_{n-k}, v_{n-k}, u_{n-k+1}, v_{n-k+1}, \dots, u_n, v_n\},$$

$$X_2 = (X_1 \setminus \{u_1, v_1\}) \cup \{u_{k+2}, v_{k+2}\},$$

$$X_3 = (X_2 \setminus \{u_n, v_n\}) \cup \{u_{n-k-1}, v_{n-k-1}\},$$

$$X_4 = (X_3 \setminus \{u_2, v_2\}) \cup \{u_{k+3}, v_{k+3}\},$$

$$X_5 = (X_4 \setminus \{u_{n-1}, v_{n-1}\}) \cup \{u_{n-k-2}, v_{n-k-2}\}, \text{ and so on.}$$

Notice that in each step we remove two elements and add two other elements. Therefore  $|X_i| = 4k + 4$  for all  $i$ . One can easily see that  $(X, T)$  is a tree decomposition for  $P(n, k)$  where  $X = \{X_1, X_2, \dots, X_{n-2k-1}\}$ . Thus,  $tw(P(n, k)) \leq 4k + 3$  and by Theorem A, the proof is complete. ■

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	
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61	60	48	59	53	58	54	57	55	56	54	55	58	54	54	53	53	56	56	53	59	50	57	5																

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